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q-deformed boson expansions

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Abstract. A deformed boson mapping of the Marumori type is derived for an underlying $su(2)$ algebra. As an example, we bosonize a pairing Hamiltonian in a two level space, for which an exact treatment is possible. Comparisons are then made between the exact result, our q-deformed boson expansion and the usual non-deformed expansion.

Nowadays, increasing importance has been given to *quantum algebraic* applications in several fields of physics [1]. In many cases, when the usual Lie algebras do not suffice to explain certain physical behaviours, quantum algebras are found to be successful mainly due to a free deformation parameter. In these cases, it is expected that a physical meaning be attached to the deformation parameter, but this is still a very challenging question. For an extensive review article on the subject, refer to [2]. In this work we are concerned with possible improvements that quantum algebras may add to boson expansions (or boson mappings).

In the literature it is easy to find situations in which fermion pairs can be replaced by bosons. This is normally performed with the help of boson mappings, that link the fermionic Hilbert space to another Hilbert space constructed with bosons. Of course boson mapping techniques are only useful when the Pauli principle effects are somehow minimized. Historically boson expansion theories were introduced from two different points of view. The first one is the Beliaev–Zelevinsky–Marshalek (BZM) method [3], which focuses on the mapping of operators by requiring that the boson images satisfy the same commutation relations as the fermion operators. In principle, all important operators can be constructed from a set of basic operators whose commutation relations form an algebra. The mapping is achieved by preserving this algebra and mapping these basic operators. The second one is the Marumori method [4], which focuses on the mapping of state vectors. This method defines the operator in such a way that the matrix elements are conserved by the mapping and the importance of the commutation rules is left as a consequence of the requirement that matrix elements coincide in both spaces. The BZM and the Marumori expansions are equivalent at infinite order, which means that just with the proper mathematics one can go from one expansion to the other.

In this paper we concentrate on this second boson mapping method. First of all, we briefly outline the main aspects of the mapping from a fermionic space to a quantum deformed bosonic space. Once the deformation parameter is set equal to one, the usual

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boson expansion is recovered. Then the simple pairing interaction model is used as an example for our calculations. The pairing Hamiltonian is exactly diagonalized and the results are compared with the ones obtained from the traditional boson and from the q-deformed boson expansions. In both cases we analyse the results for the second and fourth order Hamiltonians.

In what follows we show a Marumori type deformed boson mapping. We start from an arbitrary operator $\hat{O}$ acting on a finite fermionic space. This fermionic Hilbert space with dimension $N+1$ is spanned by a basis formed by the states $\{|n>\}$, with $n = 0, 1, ..., N$. Hence,

$$\hat{O} = \sum_{n,n'=0}^{N} \langle n'|\hat{O}|n\rangle |n'\rangle\langle n|. \tag{1}$$

In order to obtain the boson operators, we map $\hat{O} \to \hat{O}_B$:

$$\hat{O}_B = \sum_{n,n'=0}^{N} \langle n'|\hat{O}|n\rangle |n')(n| \tag{2}$$

where

$$|n) = \frac{1}{\sqrt{[n]!}} (b^\dagger)^n |0) \tag{3}$$

are the deformed boson states [5] with $[n] = \frac{q^n-1}{q-1}$ and $[b, b^\dagger]_q = bb^\dagger - qb^\dagger b = 1$. Note that the usual brackets $\langle|\rangle$ stand for fermionic states and the round brackets $(|)$ stand for bosonic states. From the above considerations, it is straightforward to check that

$$\langle m|\hat{O}|m'\rangle = (m|\hat{O}_B|m'). \tag{4}$$

Therefore, we notice that the mapping is achieved by the equality between the matrix elements in the fermionic space and their counterparts in the bosonic space. As examples, we show the expressions for the $su(2)$ operators in the deformed bosonic space:

$$(J_z)_B = \sum_{n=0}^{2j} \sum_{l=0}^{\infty} (-j+n) \frac{(-1)^l q^{l(l-1)/2}}{[n]![l]!} (b^\dagger)^{n+l} b^{n+l} \tag{5}$$

$$(J_+)_B = \sum_{n=0}^{2j} \sum_{l=0}^{\infty} \sqrt{\frac{(n+1)(2j-n)}{[n+1]}} \frac{(-1)^l q^{l(l-1)/2}}{[n]![l]!} (b^\dagger)^{n+l+1} b^{n+l} \tag{6}$$

$$(J_+J_-)_B = \sum_{n=0}^{2j} \sum_{l=0}^{\infty} n(2j-n+1) \frac{(-1)^l q^{l(l-1)/2}}{[n]![l]!} (b^\dagger)^{n+l} b^{n+l} \tag{7}$$

$$(J_-J_+)_B = \sum_{n=0}^{2j} \sum_{l=0}^{\infty} (2j-n)(n+1) \frac{(-1)^l q^{l(l-1)/2}}{[n]![l]!} (b^\dagger)^{n+l} b^{n+l} \tag{8}$$

and $(J_-)_B = (J_+)_B^\dagger$. In deducing the above expressions we have used the fact that [6]

$$|0\rangle\langle 0| =: \exp_q(-b^\dagger b) := \sum_{l=0}^{\infty} \frac{(-1)^l q^{l(l-1)/2}}{[l]!} (b^\dagger)^l b^l \tag{9}$$

and we define the $su(2)$ basis as usual, i.e. $|n\rangle = |jm\rangle$, with $m = -j+n$.

Next, we apply the q-deformed boson expansions to the pairing interaction model [7], which consists of two N-fold degenerate levels, whose energy difference is ϵ. The lower level has energy $-\epsilon/2$ and its single-particle states are usually labelled $j_1 m_1$ and the upper level has energy $\epsilon/2$ and its single-particle states are labelled $j_2 m_2$. The pairing Hamiltonian reads [8]

$$H = \frac{\epsilon}{2}\sum_m (a^\dagger_{j_1 m} a_{j_1 m} - a^\dagger_{j_2 m} a_{j_2 m}) - \frac{G}{4}\left(\sum_j \sum_m a^\dagger_{jm} a^\dagger_{j\bar{m}} \sum_{j'} \sum_{m'} a_{j'\bar{m}'} a_{j'm'} + \text{h.c.}\right) \tag{10}$$

where $a^\dagger_{j\bar{m}} = (-1)^{j-m} a_{j-m}$. In what follows, the number of particles (which are fermions) N will be even and $2j = N/2$. Introducing the quasispin $su(2)$ generators

$$S_+ = S_-^\dagger = \tfrac{1}{2}\sum_{m_1} a^\dagger_{j_1 m_1} a^\dagger_{j_1 \bar{m}_1} = \sqrt{\Omega} A_1^\dagger$$

$$S_z = \frac{1}{2}\sum_{m_1} a^\dagger_{j_1 m_1} a_{j_1 m_1} - \frac{N}{4}$$

$$L_+ = L_-^\dagger = \tfrac{1}{2}\sum_{m_2} a^\dagger_{j_2 m_2} a^\dagger_{j_2 \bar{m}_2} = \sqrt{\Omega} A_2^\dagger$$

$$L_z = \frac{1}{2}\sum_{m_2} a^\dagger_{j_2 m_2} a_{j_2 m_2} - \frac{N}{4}$$

one sees that the pairing interaction has an underlying $su(2) \otimes su(2)$ algebra. With the help of these operators, equation (10) can be rewritten as

$$H = \epsilon(S_z - L_z) - \frac{G\Omega}{2}((A_1^\dagger + A_2^\dagger)(A_1 + A_2) + (A_1 + A_2)(A_1^\dagger + A_2^\dagger)). \tag{11}$$

The basis of states used for the diagonalization of the above Hamiltonian is $|S = \frac{N}{4} L_z, L = \frac{N}{4} - L_z\rangle$ [7, 9].

Deformation can be straightforwardly introduced by deforming the $su(2)\otimes su(2)$ algebra and this problem has already been tackled in [9]. To check the validity of the boson expansion method proposed in this letter, we substitute equations (5)–(8) into equation (11) and obtain for the fourth order Hamiltonian:

$$\begin{aligned}\frac{H_4}{\epsilon} = -\frac{x}{2} &+ \left(1 - \frac{x(\Omega-1)}{2\Omega}\right) b_1^\dagger b_1 + \left(-1 - \frac{x(\Omega-1)}{2\Omega}\right) b_2^\dagger b_2 - \frac{x}{2}(b_1^\dagger b_2 + b_2^\dagger b_1) \\ &+ \left(\frac{2}{[2]} - 1\right)(b_1^\dagger b_1^\dagger b_1 b_1 - b_2^\dagger b_2^\dagger b_2 b_2) \frac{-x}{4\Omega}\left(2 - 3\Omega - \frac{8}{[2]} + \frac{5\Omega}{[2]} + \frac{\Omega}{[2]} q\right) \\ &\times (b_1^\dagger b_1^\dagger b_1 b_1 + b_2^\dagger b_2^\dagger b_2 b_2) - \frac{x}{2\Omega}\left(\sqrt{\frac{2\Omega(\Omega-1)}{[2]}} - \Omega\right) \\ &\times (b_1^\dagger b_2^\dagger b_2 b_2 + b_1^\dagger b_1^\dagger b_1 b_2 + \text{h.c.})\end{aligned} \tag{12}$$

where $x = 2G\Omega/\epsilon$. The second order Hamiltonian is easily read off from the above equation by omitting all terms containing four boson operators. Diagonalizing equation (12) is a simple task and for this purpose the basis used is

$$|n_1 n_2) = \frac{1}{\sqrt{[n_1]![n_2]!}} (b_1^\dagger)^{n_1} (b_2^\dagger)^{n_2} |0) \tag{13}$$

and

$$b_1^\dagger |n_1) = \sqrt{[n_1 + 1]} |n_1 + 1) \qquad b_1 |n_1) = \sqrt{[n_1]} |n_1 - 1)$$

with similar expressions for the b_2 and $b_2^\dagger$ operators. We finally obtain:

$$\frac{H_4}{\epsilon}|n_1 n_2) = \left(-\frac{x}{2} + \left(\frac{2}{[2]} - 1\right)([n_1][n_1 - 1] - [n_2][n_2 - 1]) + \left(1 - \frac{x(\Omega-1)}{2\Omega}\right)[n_1]\right.$$

$$
\begin{aligned}
&+\left(-1-\frac{x(\Omega-1)}{2\Omega}\right)[n_2]\frac{-x}{4\Omega}\left(2-3\Omega-\frac{8}{[2]}+\frac{5\Omega}{[2]}+\frac{\Omega}{[2]}q\right)\\
&\times([n_1][n_1-1]+[n_2][n_2-1])\Bigg)|n_1n_2)\\
&+\left(-\frac{x}{2}\sqrt{[n_2][n_1+1]}-\frac{x}{2\Omega}\left(\sqrt{\frac{2\Omega(\Omega-1)}{[2]}}-\Omega\right)\right.\\
&\left.\times([n_2-1]+[n_1])\sqrt{[n_2][n_1+1]}\right)|n_1+1n_2-1)\\
&+\left(-\frac{x}{2}\sqrt{[n_1][n_2+1]}-\frac{x}{2\Omega}\left(\sqrt{\frac{2\Omega(\Omega-1)}{[2]}}-\Omega\right)\right.\\
&\left.\times([n_1-1]+[n_2])\sqrt{[n_2+1][n_1]}\right)|n_1-1\;n_2+1).
\end{aligned}
\tag{14}
$$

Equation (14) yields the energy spectrum for the deformed Marumori type boson expansion. When q is set equal to unity, the non-deformed spectrum is obtained. In what follows, we have chosen $x = 1.0$ and the degeneracy $\Omega = 20$. In figure 1 we show the ground state energy resulting from the exact diagonalization of equation (11) and the ground state energies obtained from the second and fourth order Hamiltonians defined in equation (12) as a function of the number of pairs for $q = 1$. One can see that the fourth order curve lies closer to the exact result than the second order curve, as expected, once the full expansion converges to the exact result.

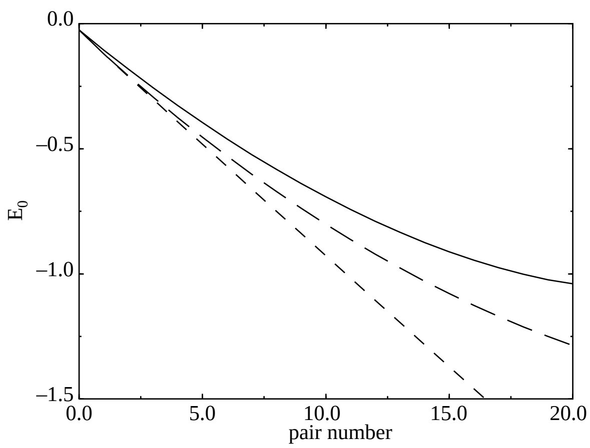

Figure 1. The ground state energy E_0 is plotted as a function of the number of pairs for the exact result (full curve), the second order expansion result (short-dashed curve) and for the fourth order result (long-dashed curve) for $q = 1$, the interaction strength $x = 1.0$ and the degeneracy $\Omega = 20$.

We then compare the exact result with the deformed second and fourth order expansions and the results are plotted in figure 2. For those cases, the parameter q was chosen in order to fit the exact result and we find that the second order expansion converges if we set $q = 0.862$ and the fourth order expansion also converges if $q = 0.810$. This implies that the deformation parameter is playing the same rôle as all the rest of the truncated expansion

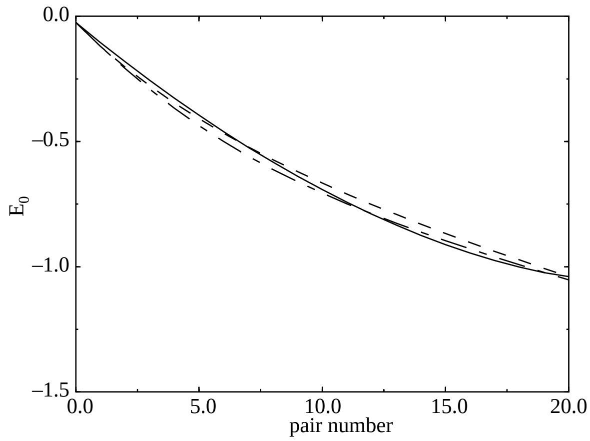

Figure 2. The ground state energy E_0 is plotted as a function of the number of pairs for the exact result with $q = 1$ (full curve), the second order expansion result with $q = 0.862$ (broken curve) and for the fourth order result with $q = 0.810$ (chain curve) for the interaction strength $x = 1.0$ and the degeneracy $\Omega = 20$.

and one does not have to go beyond the deformed second order boson expansion to obtain the exact result, while the fourth order non-deformed expansion still gives very poor results, as seen in figure 1.

As is well known, boson expansion theories are usually applied to problems where the exact fermionic results are either difficult or impossible to obtain and the boson expansion methods are useful whenever the expansion converges rapidly. The quantum deformation parameter can be used to accelerate this convergence and, in this work, we have chosen a model for which the exact result is known in order to verify that point. In this respect, one should bear in mind that the q-deformed boson expansion is unitarily equivalent to the usual boson theories, i.e. at infinite order they are identical. Therefore, the use of quantum algebras in boson expansion theory can be a very useful method in providing the same result as the complete series. In cases where experimental results are known, q has also been used in the literature for achieving the best fitting, as in [10]. We may then conclude that further investigations, like the consideration of the BZM method and also of other model Hamiltonians, deserve some effort in the future.

Acknowledgments

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